STRUCTURE OF THERMAL STRESSES ARISING IN A VISCOELASTIC HALF-SPACE WITH THERMAL "MEMORY"

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We develop the structure of thermal stresses arising in a viscoelastic half-space owing to the thermal impact of a heat flux at the boundary.

We consider a viscoelastic half-space, without an external load, with thermal "memory," subjected to sudden heating by a heat flux along the boundary surface z = 0. In order to determine the temperature field in the half-space we use the Gurtin–Pipkin heat equation [1]

$$c_v\ddot{\theta} + \beta(0)\dot{\theta} + \int_0^\infty \beta'(s)\dot{\theta}(t-s)\,ds = \alpha(0)\theta'' + \int_0^\infty \alpha'(s)\theta''(t-s)\,ds \tag{1}$$

with the boundary conditions

$$\theta = \dot{\theta} = 0 \quad \text{for} \quad t = 0, \tag{2}$$

$$-\int_0^\infty \alpha(s)\left.\frac{\partial\theta(z,\ t-s)}{\partial z}\right|_{z=0} ds = q_0\eta(t); \quad \theta|_{z\to\infty} = 0. \tag{3}$$

Here $\eta(t)$ is the Heaviside unit function

According to [2], the Laplace transforms of the thermal stresses in a viscoelastic half-space have the form

$$\bar{\sigma}_{zz} = p^2\rho\bar{\Phi}; \quad \bar{\sigma}_{xx} = \bar{\sigma}_{yy} = -2\bar{m}\bar{\mu}\,\bar{\theta} + \bar{\lambda}\,\bar{\sigma}^2 p^2\bar{\Phi}, \tag{4}$$

where Φ is the Laplace transform of the retarded thermoelastic potential of the displacements, which satisfies the equation

$$\frac{d^2\bar{\Phi}}{dz^2} - p^2\bar{\sigma}^2\bar{\Phi} = \bar{m}\,\bar{\theta}, \tag{5}$$

where $\bar{m} = \bar{\gamma}/(\bar{\lambda} + 2\bar{\mu})$; $\bar{\sigma}^2 = \rho/(\bar{\lambda} + 2\bar{\mu})$; and λ, μ, and γ are functions characterizing the rheological properties of the medium.

Solving the boundary-value problem (1)-(3) and substituting the temperature transform into (5), we find the form of $\bar{\Phi}$ that satisfies Eq. (5):

$$\bar{\Phi} = A_1\exp(-p\bar{\sigma}z) + A_2\exp(p\bar{\sigma}z) + \frac{q_0\bar{m}\exp\left(-\sqrt{p\,\frac{c_v+\bar{\beta}}{\bar{\alpha}}}\,z\right)}{p^{5/2}\sqrt{\bar{\alpha}(c_v+\bar{\beta})}\left(\frac{c_v+\bar{\beta}}{\bar{\alpha}} - \bar{\sigma}^2 p\right)}. \tag{6}$$

Since the half-space is not loaded, we have that

$$\bar{\sigma}_{zz}\Big|_{\substack{z=0\\ z\to\infty}} = 0; \quad \bar{\Phi}\Big|_{\substack{z=0\\ z\to\infty}} = 0. \tag{7}$$

Determining the constants of integration from conditions (7), we finally obtain

$$\bar{\Phi} = \frac{q_0\bar{m}\left\{\exp\left[-z\sqrt{p\,\frac{c_v+\bar{\beta}}{\bar{\alpha}}}\right] - \exp\left[-\bar{\sigma}pz\right]\right\}}{p^{5/2}\sqrt{\bar{\alpha}(c_v+\bar{\beta})}\left(\frac{c_v+\bar{\beta}}{\bar{\alpha}} - \bar{\sigma}^2 p\right)}, \tag{8}$$

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$$\bar{\theta} = \frac{q_0 \exp\left[-z\sqrt{p\frac{c_v+\bar{\beta}}{\bar{\alpha}}}\right]}{p^{3/2}\sqrt{\bar{\alpha}(c_v+\bar{\beta})}}. \tag{9}$$

Proceeding from Eqs. (8) and (9), we construct the structure of the thermal stresses for a viscoelastic half-space provided by thermal "memory" (Fig. 1).

In the structure of the thermal stresses there enter the transforms of the functions $\bar{\lambda}$, $\bar{\mu}$, and $\bar{\gamma}$, which are given by the model of the behavior of the viscoelastic medium. For a Biot material they have the form

$$\bar{\mu} = p\bar{a}, \quad \bar{\lambda} = p\bar{b}, \quad \bar{\gamma} = \alpha_t p(3\bar{b} + 2\bar{a}). \tag{10}$$

We represent the functions a and b that characterize the rheological properties of the medium in the form

$$a(t) = \mu_0 \exp(-\varepsilon t), \tag{11}$$

$$b(t) = \lambda_0 \exp(-\varepsilon t). \tag{12}$$

Then

$$\bar{\mu} = \mu_0 \frac{p}{p+\varepsilon}; \quad \lambda = \lambda_0 \frac{p}{p+\varepsilon}; \quad \bar{\sigma}^2 = \sigma_0^2 \frac{p+\varepsilon}{p};$$
$$\sigma_0^2 = \frac{\rho}{\lambda_0 + 2\mu_0}; \quad \gamma_0 = \alpha_t(3\lambda_0 + 2\mu_0); \quad \bar{m} = m_0 = \gamma_0\sigma_0^2/\rho. \tag{13}$$

We represent the functions of relaxation of the heat flux and the internal energy for small intervals of time in the form [3]

$$\alpha(t) = \alpha(0) + \alpha'(0)t + o(t^2), \tag{14}$$

$$\beta(t) = \beta(0) + \beta'(0)t + o(t^2). \tag{15}$$

Substituting (13) and the transforms (14) and (15) into (8) and (9), and expanding the expressions $(c_V + \bar{\beta})/\bar{\alpha}$ and $[(c_V + \bar{\beta})/\bar{\alpha}]^{1/2}$ in Taylor series with accuracy to terms of order of smallness 1/p, we obtain

$$\bar{\theta} \approx \frac{q_0 \exp\left[-p\frac{z}{u} - z\frac{\xi^-}{2u}\right]}{pc_v u}\left(1 - \frac{\xi^+}{2u}\right), \tag{16}$$

$$\bar{\sigma}_{zz} \approx \frac{q_0\gamma_0\sigma_0^2\left\{\exp\left[-p\frac{z}{u} - z\frac{\xi^-}{2u}\right] - \exp\left[-z\sigma_0\sqrt{p(p+\varepsilon)}\right]\right\}}{pc_v u(u^{-2} - \sigma_0^2)}\left[1 - \frac{1}{p}\left(\omega + \frac{\xi^+}{2}\right)\right]. \tag{17}$$

In this case the complete transfer function of the structural scheme (Fig. 1) can be transformed. We obtain the reaction of the output signal, the normal stresses for a sudden impact, by a heat flux along the surface of the half-space:

$$\sigma_{zz} \approx \frac{q_0\gamma_0\sigma_0^2}{c_v u(u^{-2} - \sigma_0^2)}\left\{\exp\left(-\frac{z\xi^-}{2u}\right)\eta\left(t - \frac{z}{u}\right)\left[1 - \left(\omega + \frac{\xi^+}{2}\right) \times \right.\right.$$
$$\left.\left. \times \left(t - \frac{z}{u}\right)\right] - \eta(t - z\sigma_0)\left[\varphi(z, t) - \left(\omega + \frac{\xi^+}{2}\right)\int_0^t \varphi(z, \xi)\,d\xi\right]\right\}, \tag{18}$$

where

$$\varphi(z, t) = e^{-\frac{\varepsilon\sigma_0}{2}z} + \frac{\varepsilon\sigma_0 z}{2}\int_{z\sigma_0}^{t} e^{-\frac{\varepsilon}{2}\xi}\frac{I_1\left[\frac{\varepsilon}{2}\sqrt{\xi^2 - z^2\sigma_0^2}\right]}{\sqrt{\xi^2 - z^2\sigma_0^2}}\,d\xi;$$

$$u = \sqrt{\frac{\alpha(0)}{c_v}}; \quad \omega = \frac{\frac{\xi^-}{u^2} - \sigma_0^2\varepsilon}{u^{-2} - \sigma_0^2}; \quad \xi^\pm = \frac{\beta(0)}{c_v} \pm \frac{\alpha'(0)}{\alpha(0)}.$$

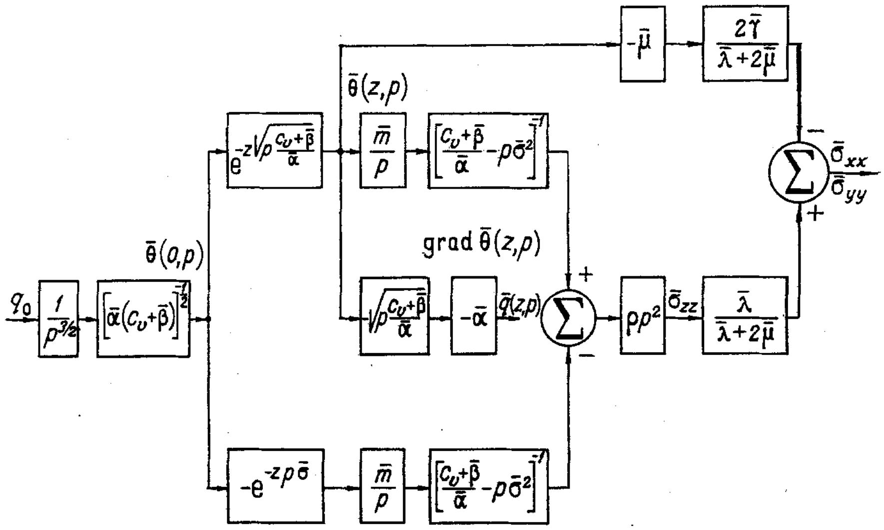

Fig. 1. Structure of the thermal stresses in a viscoelastic half-space with thermal "memory" arising owing to the thermal impact by a heat flux at the boundary.

We see that owing to the thermal impact in the half-space, two waves of thermal stresses begin to propagate: the first wave with velocity of thermal perturbations in the Gurtin–Pipkin theory $u = [\alpha(0)/c_V]^{1/2}$, the second with velocity $\sigma_0^{-1} = [(\lambda_0 + 2\mu_0)/\rho]^{1/2}$, i.e., with the velocity of the longitudinal elastic vibrations. On the wavefronts the stress undergoes a discontinuity

$$[\sigma_{zz}]|_{t=z/u} = \frac{q_0\gamma_0\sigma_0^2}{c_v u\,(u^{-2}-\sigma_0^2)}\,e^{-z\xi^{-}/2u};\quad [\sigma_{zz}]|_{t=z/\sigma_0^{-1}} = \frac{q_0\gamma_0\sigma_0^2 e^{-\varepsilon z\sigma_0/2}}{c_v u\,(u^{-2}-\sigma_0^2)}. \qquad (19)$$

Formation of the two stress waves is reflected in the structural scheme by the two branches. Giving, in a different way, the functions characterizing the rheological behavior of the medium and the relaxation function of the heat flux and the internal energy, we obtain structures of the thermal stresses for various rheological materials and various models of thermal "memory." Thus, if relaxation of the internal energy does not occur ($\beta = 0$), and the relaxation function of the heat flux is given in the form of the Maxwell–Cattaneo core $\alpha = \frac{\lambda}{\tau_r}\exp[-\frac{t}{\tau_r}]$, we obtain the structure of the thermal stresses in generalized thermomechanics [4].

On the basis of the operator relations between separate observable physical coordinates on the structure of viscoelastic thermal stresses (Fig. 1) we can give a program of the experiment for determining the functions that characterize the rheological behavior of the medium and the thermal "memory," i.e., we can determine the real model of behavior of a viscoelastic medium with thermal "memory."

NOTATION

z, coordinate normal to the surface of the half-space; t, time; $\theta = T - T_0$, temperature of the half-space; $\alpha(t)$, relaxation function of the heat flux; $\beta(t)$, relaxation function of the internal energy; c_V, specific heat at constant volume; q_0, heat flux acting on the boundary of the half-space; σ_{xx}, σ_{yy}, σ_{zz}, normal stresses; ρ, density of the material; α_t, coefficient of linear thermal expansion; $u = [\alpha(0)/c_V]^{1/2}$, heat-propagation velocity; λ_t, coefficient of thermal conductivity; τ_r, relaxation time of the heat flux; ε, relaxation time of the stresses.

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